

# Inhomogeneous fluids with density-dependent viscosity

L. Fusi<sup>1</sup>   A. Farina<sup>1</sup>   F. Rosso<sup>1</sup>   K.R. Rajagopal<sup>2</sup>

<sup>1</sup>Department of Mathematics and Informatics "U. Dini",  
University of Florence, Italy

<sup>2</sup>Dept. Mech. Engineering, Texas A& M University, College Station U.S.

Quito, 22-26 october 2018,

XVI ENCUENTRO DE MATEMATICA Y SUS APLICACIONES

# Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity
- 3 The initial-boundary value problem
- 4 Numerics
- 5 Analytic solutions
- 6 Comparison between the analytic solution and the numerical solution

## Apparently homogeneous materials may be far from that

- **Perfect homogeneous bodies do not really exist.** Inhomogeneity is **often ignored** just to simplify the analysis. However this simplification is allowable *if and only if* compressibility has *insignificant* consequences concerning the “response functional” (which identifies the Cauchy stress tensor).
- Many **important fluid** (i.e. biological, geological, industrial) are commonly treated as homogeneous although they are very far from that. **Their dynamical behaviour cannot be understood without giving up to the homogeneity approximation.**
- **Even more important:** giving up to inhomogeneity may be a **tremendous source of errors** (examples by Anand & Rajagopal (2005)).
- **Consequence:** we need a more **careful approach** to identify situations which allow the fluid to be treated as homogeneous and those which cannot.

## Apparently homogeneous materials may be far from that

- **Perfect homogeneous bodies do not really exist.** Inhomogeneity is **often ignored** just to simplify the analysis. However this simplification is allowable *if and only if* compressibility has *insignificant* consequences concerning the “response functional” (which identifies the Cauchy stress tensor).
- Many **important fluid** (i.e. biological, geological, industrial) are commonly treated as homogeneous although they are very far from that. **Their dynamical behaviour cannot be understood without giving up to the homogeneity approximation.**
- **Even more important:** giving up to inhomogeneity may be a **tremendous source of errors** (examples by Anand & Rajagopal (2005)).
- **Consequence:** we need a more **careful approach** to identify situations which allow the fluid to be treated as homogeneous and those which cannot.

## Apparently homogeneous materials may be far from that

- **Perfect homogeneous bodies do not really exist.** Inhomogeneity is **often ignored** just to simplify the analysis. However this simplification is allowable *if and only if* compressibility has *insignificant* consequences concerning the “response functional” (which identifies the Cauchy stress tensor).
- Many **important fluid** (i.e. biological, geological, industrial) are commonly treated as homogeneous although they are very far from that. **Their dynamical behaviour cannot be understood without giving up to the homogeneity approximation.**
- **Even more important:** giving up to inhomogeneity may be a **tremendous source of errors** (examples by Anand & Rajagopal (2005)).
- **Consequence:** we need a more **careful approach** to identify situations which allow the fluid to be treated as homogeneous and those which cannot.

## Apparently homogeneous materials may be far from that

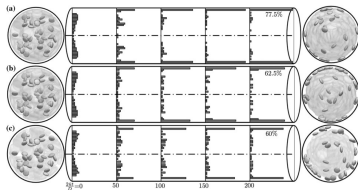
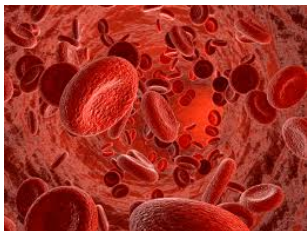
- **Perfect homogeneous bodies do not really exist.** Inhomogeneity is **often ignored** just to simplify the analysis. However this simplification is allowable *if and only if* compressibility has *insignificant* consequences concerning the “response functional” (which identifies the Cauchy stress tensor).
- Many **important fluid** (i.e. biological, geological, industrial) are commonly treated as homogeneous although they are very far from that. **Their dynamical behaviour cannot be understood without giving up to the homogeneity approximation.**
- **Even more important:** giving up to inhomogeneity may be a **tremendous source of errors** (examples by Anand & Rajagopal (2005)).
- **Consequence:** we need a more **careful approach** to identify situations which allow the fluid to be treated as homogeneous and those which cannot.

## Apparently homogeneous materials may be far from that

- **Perfect homogeneous bodies do not really exist.** Inhomogeneity is **often ignored** just to simplify the analysis. However this simplification is allowable *if and only if* compressibility has *insignificant* consequences concerning the “response functional” (which identifies the Cauchy stress tensor).
- Many **important fluid** (i.e. biological, geological, industrial) are commonly treated as homogeneous although they are very far from that. **Their dynamical behaviour cannot be understood without giving up to the homogeneity approximation.**
- **Even more important:** giving up to inhomogeneity may be a **tremendous source of errors** (examples by Anand & Rajagopal (2005)).
- **Consequence:** we need a more **careful approach** to identify situations which allow the fluid to be treated as homogeneous and those which cannot.

## Blood: a “classical” (but very complex) inhomogeneous fluid

This complexity was very well investigated by Fasano and Sequeira (HEMOMATH: the mathematics of blood, 2017)

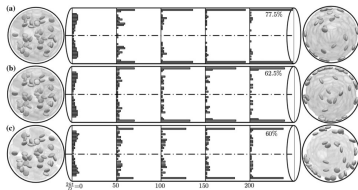
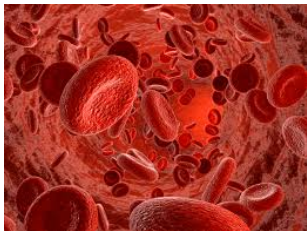


The mechanism of platelets “margination” may guide us to identify risk factors associated with atherothrombosis. Due to the small time and length scales and optical properties of blood, the platelets dynamics near the vessel is a real challenging problem.

**Notice:** *blood is generally treated as incompressible and is manifestly inhomogeneous*

## Blood: a “classical” (but very complex) inhomogeneous fluid

This complexity was very well investigated by Fasano and Sequeira (HEMOMATH: the mathematics of blood, 2017)

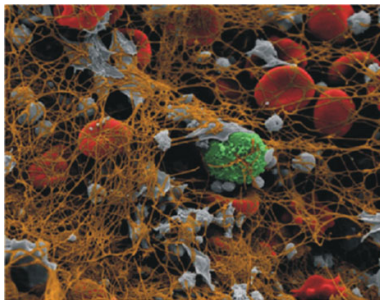


The mechanism of platelets “margination” may guide us to identify risk factors associated with atherothrombosis. Due to the small time and length scales and optical properties of blood, the platelets dynamics near the vessel is a real challenging problem.

**Notice:** *blood is generally treated as incompressible and is manifestly inhomogeneous*

## Blood: a “classical” (but very complex) inhomogeneous fluid

A micro-photography of a forming clot shows clearly the inhomogeneity of blood



In this case various components are taken into account through their concentrations.

## A “geological” example: landslides, mudslides (typical granular materials)



Figure: The Grand Mesa (Colorado, USA) landslide

*Notice: in particular situation (interlocking, small grains,...) granular materials can be considered incompressible. Most geological granular materials are inhomogeneous*

## A “geological” example: landslides, mudslides (typical granular materials)



Figure: The Grand Mesa (Colorado, USA) landslide

*Notice: in particular situation (interlocking, small grains,...) granular materials can be considered incompressible. Most geological granular materials are inhomogeneous*

## A “geological” example: landslides, mudslides (typical granular materials)



Figure: The Grand Mesa (Colorado, USA) landslide

**Notice:** *in particular situation (interlocking, small grains,...) granular materials can be considered incompressible. Most geological granular materials are inhomogeneous*

## Another example: polymers

Macroscopic *inhomogeneity* normally associated with the history of processing: when two homopolymers, say A and B, are blended in the melt state or cast from a common solvent, their chemical differences are normally sufficient to induce a liquid-liquid phase separation

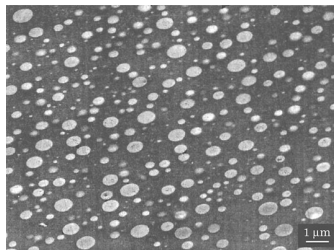


Figure: An 80/20 composition polyethersulfone/ polycarbonate blend. Reproduced from Weber (1999)

**Notice:** a polymer melt is treated as *incompressible* both before and after cooling

## Another example: polymers

Macroscopic *inhomogeneity* normally associated with the history of processing: when two homopolymers, say A and B, are blended in the melt state or cast from a common solvent, their chemical differences are normally sufficient to induce a liquid-liquid phase separation

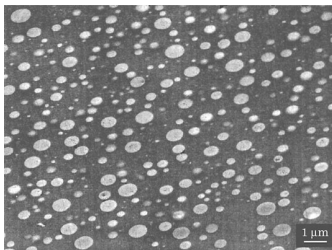


Figure: An 80/20 composition polyethersulfone/ polycarbonate blend. Reproduced from Weber (1999)

**Notice:** a polymer melt is treated as *incompressible* both before and after cooling

## Historical approaches to inhomogeneous materials

### Korteweg's model (1901)

$$\mathbb{T} = (-p + \alpha_0 |\nabla \varrho|^2 + \alpha_1 \nabla^2 \varrho) \mathbb{I} + \beta (\nabla \varrho \otimes \nabla \varrho) + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

$\mathbb{T}$  (symmetric) stress tensor,  $\mathbf{v}$  particle velocity,  $p(\varrho) =$  thermodynamic pressure,

$\mathbb{D}(\mathbf{v}) =$  symmetric part of the velocity gradient,  $\alpha_i, \lambda, \beta, \mu$  are material scalars **that may depend on  $\varrho$  as well**.  $\Rightarrow \nabla \cdot \mathbb{T}$  becomes highly non linear!

The above form should be compared with the standard Navier-Stokes model for compressible fluids:

$$\mathbb{T}^{\text{ns}} = -p \mathbb{I} + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

However it was not clear if the Korteweg's model is consistent with the principles of thermodynamics. Dunn and Serrin (1985) (and many others since then) investigated this issue. Their approach however shows some **inconsistencies** that can be overcome only by introducing some **questionable concepts** as "interstitial working" or "micropolarity".

## Historical approaches to inhomogeneous materials

### Korteweg's model (1901)

$$\mathbb{T} = (-p + \alpha_0 |\nabla \varrho|^2 + \alpha_1 \nabla^2 \varrho) \mathbb{I} + \beta (\nabla \varrho \otimes \nabla \varrho) + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

$\mathbb{T}$  (symmetric) stress tensor,  $\mathbf{v}$  particle velocity,  $p(\varrho) =$  thermodynamic pressure,

$\mathbb{D}(\mathbf{v}) =$  symmetric part of the velocity gradient,  $\alpha_i, \lambda, \beta, \mu$  are material scalars **that may depend on  $\varrho$  as well**.  $\Rightarrow \nabla \cdot \mathbb{T}$  becomes highly non linear!

The above form should be compared with the standard Navier-Stokes model for compressible fluids:

$$\mathbb{T}^{\text{ns}} = -p \mathbb{I} + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

However it was not clear if the Korteweg's model is consistent with the principles of thermodynamics. Dunn and Serrin (1985) (and many others since then) investigated this issue. Their approach however shows some **inconsistencies** that can be overcome only by introducing some **questionable concepts** as "interstitial working" or "micropolarity".

## Historical approaches to inhomogeneous materials

### Korteweg's model (1901)

$$\mathbb{T} = (-p + \alpha_0 |\nabla \varrho|^2 + \alpha_1 \nabla^2 \varrho) \mathbb{I} + \beta (\nabla \varrho \otimes \nabla \varrho) + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

$\mathbb{T}$  (symmetric) stress tensor,  $\mathbf{v}$  particle velocity,  $p(\varrho) =$  thermodynamic pressure,

$\mathbb{D}(\mathbf{v}) =$  symmetric part of the velocity gradient,  $\alpha_i, \lambda, \beta, \mu$  are material scalars **that may depend on  $\varrho$  as well**.  $\Rightarrow \nabla \cdot \mathbb{T}$  becomes highly non linear!

The above form should be compared with the standard Navier-Stokes model for compressible fluids:

$$\mathbb{T}^{\text{ns}} = -p \mathbb{I} + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

However it was not clear if the Korteweg's model is consistent with the principles of thermodynamics. Dunn and Serrin (1985) (and many others since then) investigated this issue. Their approach however shows some **inconsistencies** that can be overcome only by introducing some **questionable concepts** as "interstitial working" or "micropolarity".

## Historical approaches to inhomogeneous materials

### Korteweg's model (1901)

$$\mathbb{T} = (-p + \alpha_0 |\nabla \varrho|^2 + \alpha_1 \nabla^2 \varrho) \mathbb{I} + \beta (\nabla \varrho \otimes \nabla \varrho) + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

$\mathbb{T}$  (symmetric) stress tensor,  $\mathbf{v}$  particle velocity,  $p(\varrho) =$  thermodynamic pressure,

$\mathbb{D}(\mathbf{v}) =$  symmetric part of the velocity gradient,  $\alpha_i, \lambda, \beta, \mu$  are material scalars **that may depend on  $\varrho$  as well**.  $\Rightarrow \nabla \cdot \mathbb{T}$  becomes highly non linear!

The above form should be compared with the standard Navier-Stokes model for compressible fluids:

$$\mathbb{T}^{\text{ns}} = -p \mathbb{I} + 2\mu \mathbb{D}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{v}) \mathbb{I}$$

However it was not clear if the Korteweg's model is consistent with the principles of thermodynamics. Dunn and Serrin (1985) (and many others since then) investigated this issue.

Their approach however shows some **inconsistencies** that can be overcome only by introducing some **questionable concepts** as “interstitial working” or “micropolarity”.

## A more recent approach (Rajagopal and Srinivasa (2008))

Use the theory of “multiple natural configurations” (Eckart (1948), Rajagopal (1995), Rajagopal and Srinivasa (2000)): a unified frame work for rate type fluids.

- Theory of multiple configurations allows to derive reasonable constitutive equations of a large class of dissipative materials.
- Central idea: a body can exist in multiple stress-free configurations whose evolution is regulated by thermodynamics.
- The evolution of the natural configuration is established by the maximization of the rate of entropy production. *For the isothermal case this is equivalent to the maximization of the rate of dissipation.*

## A more recent approach (Rajagopal and Srinivasa (2008))

Use the theory of “multiple natural configurations” (Eckart (1948), Rajagopal (1995), Rajagopal and Srinivasa (2000)): a unified frame work for rate type fluids.

- Theory of multiple configurations allows to derive reasonable constitutive equations of a large class of dissipative materials.
- Central idea: a body can exist in multiple stress-free configurations whose evolution is regulated by thermodynamics.
- The evolution of the natural configuration is established by the maximization of the rate of entropy production. *For the isothermal case this is equivalent to the maximization of the rate of dissipation.*

## A more recent approach (Rajagopal and Srinivasa (2008))

Use the theory of “multiple natural configurations” (Eckart (1948), Rajagopal (1995), Rajagopal and Srinivasa (2000)): a unified frame work for rate type fluids.

- Theory of multiple configurations allows to derive reasonable constitutive equations of a large class of dissipative materials.
- Central idea: a body can exist in multiple stress-free configurations whose evolution is regulated by thermodynamics.
- The evolution of the natural configuration is established by the maximization of the rate of entropy production. *For the isothermal case this is equivalent to the maximization of the rate of dissipation.*

## A more recent approach (Rajagopal and Srinivasa (2008))

Use the theory of “multiple natural configurations” (Eckart (1948), Rajagopal (1995), Rajagopal and Srinivasa (2000)): a unified frame work for rate type fluids.

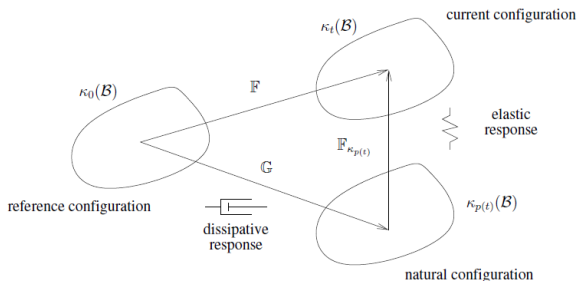
- Theory of multiple configurations allows to derive reasonable constitutive equations of a large class of dissipative materials.
- Central idea: a body can exist in multiple stress-free configurations whose evolution is regulated by thermodynamics.
- The evolution of the natural configuration is established by the maximization of the rate of entropy production. ***For the isothermal case this is equivalent to the maximization of the rate of dissipation.***

## Key idea

Deformation gradient  $\mathbb{F} \stackrel{\text{def}}{=} \text{Grad } \varphi(\mathbf{X}, t)$

Natural configuration

- splits the deformation  $\mathbb{F}$  into the elastic and dissipative parts  $\mathbb{F}_{\kappa_P(t)}$  and  $\mathbb{G}$



- $$\mathbb{F} = \mathbb{F}_{\kappa_P(t)} \mathbb{G}$$

# Fundamental equations of Continuum Mechanics

**Mass conservation:**

$$\boxed{\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0} \quad \text{where} \quad (\dot{\bullet}) = \frac{\partial (\bullet)}{\partial t} + \mathbf{v} \cdot \nabla (\bullet)$$

**Governing equations:** linear and angular momentum

$$\boxed{\rho \dot{\mathbf{v}} = \nabla \cdot \mathbb{T} + \rho \mathbf{f}} \quad \boxed{\mathbb{T}^T = \mathbb{T}}$$

**Internal energy balance**

$$\boxed{\rho \dot{e} = \mathbb{T} : \mathbb{D} - \nabla \cdot \mathbf{q} + \rho r}$$

**Clausius-Duhem inequality**

$$\boxed{\rho \zeta \stackrel{\text{def}}{=} \rho \dot{\eta} + \nabla \cdot \left( \frac{\mathbf{q}}{\vartheta} \right) - \frac{\rho r}{\vartheta}, \quad \zeta \geq 0} \quad (\zeta \text{ rate of entropy production per unit mass})$$

**For isothermal processes:** define  $\Psi \stackrel{\text{def}}{=} e - \vartheta_0 \eta \geq 0$  (Helmholtz potential) and  $\xi \stackrel{\text{def}}{=} \rho \vartheta_0 \zeta$ . Then

$$\mathbb{T} : \mathbb{D} - \rho \dot{\Psi} = \xi \geq 0$$

## Reduced model for isothermal processes of an incompressible body at a given uniform temperature

$$\operatorname{tr} \mathbb{D} = \nabla \cdot \mathbf{v} = 0, \quad \dot{\varrho} = \frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \varrho = 0$$

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla_{\mathbf{x}} \mathbf{v}] \mathbf{v} \right) = \nabla \cdot \mathbb{T} + \varrho \mathbf{f}$$

where the constitutive quantities  $\mathbb{T}$ ,  $\xi$ ,  $\Psi$  satisfy

$$\mathbb{T} : \mathbb{D} - \varrho \dot{\Psi} = \xi, \quad \xi \geq 0, \quad \Psi \geq 0$$

NOTICE: from the “material” point of view the second equation reads as

$$\varrho = \tilde{\varrho}(\mathbf{x}, t) = \tilde{\varrho}(\chi(\mathbf{X}, t), t) = \hat{\varrho}(\mathbf{X}, t) = \hat{\varrho}(\mathbf{X}, 0) \stackrel{\text{def}}{=} \varrho_R(\mathbf{X}).$$

i.e. the density of a fixed particle does not change although the density can change from point to point in the “abstract” body. **This is what we mean in Continuum Mechanics with “incompressible and inhomogeneous”.**

## Generic form of the Cauchy stress tensor for inhomogeneous materials (Malek et al. 2006)

Use the “natural configurations approach” and the “maximization principle” of the entropy production for deriving reasonable constitutive relations for  $\mathbb{T}$  (Rajagopal,1995) (Rajagopal and Srinisava, 2000):

$$\mathbb{T} = -p\mathbb{I} - \varrho \left( \frac{\partial \Psi}{\partial(\nabla p)} \otimes \nabla \varrho - \frac{1}{3} \frac{\partial \Psi}{\partial(\nabla \varrho)} \cdot \nabla \varrho \mathbb{I} \right) + 2\mu(p, \varrho, \mathbb{D}^2)\mathbb{D}.$$

where  $\mu$  has the meaning (and dimension) of a “viscosity”.

**The specification of  $\psi$  (i.e. the way energy is stored in the continuum) and  $\mu$  (i.e. the way the system dissipates energy), means to know the “response function”.**

## Generic form of the Cauchy stress tensor for inhomogeneous materials (Malek et al. 2006)

Use the “natural configurations approach” and the “maximization principle” of the entropy production for deriving reasonable constitutive relations for  $\mathbb{T}$  (Rajagopal, 1995) (Rajagopal and Srinisava, 2000):

$$\mathbb{T} = -p\mathbb{I} - \varrho \left( \frac{\partial \Psi}{\partial(\nabla p)} \otimes \nabla \varrho - \frac{1}{3} \frac{\partial \Psi}{\partial(\nabla \varrho)} \cdot \nabla \varrho \mathbb{I} \right) + 2\mu(p, \varrho, \mathbb{D}^2)\mathbb{D}.$$

where  $\mu$  has the meaning (and dimension) of a “viscosity”.

**The specification of  $\psi$  (i.e. the way energy is stored in the continuum) and  $\mu$  (i.e. the way the system dissipates energy), means to know the “response function”.**

## (Anand & Rajagopal, 2005) Forgetting inhomogeneity can be a tremendous source of errors!

Consider the following example (isothermal, power law fluid): inhomogeneity in  $\rho$  leads to inhomogeneity in the viscosity (in a known way)

$$\mathbb{T} = -p\mathbb{I} + \mu(\mathbf{X}) \left( 1 + \beta \left( \frac{1}{2} \text{tr} \mathbb{D}^2 \right) \right)^m \mathbb{D}$$

$\mathbb{D}$  is the symmetric part of the velocity gradient (for  $m = 0$  we get a Navier-Stokes inhomogeneous fluid),  $\beta =$  positive constant.

**Plane Poiseuille flow** (Lagrangian approach):

$$x = X + u(Y)t, \quad y = Y, \quad z = Z$$

or **cylindrical Couette**:

$$r = R, \quad \vartheta = \Theta + \omega(R)t, \quad z = Z$$

## (Anand & Rajagopal, 2005) Forgetting inhomogeneity can be a tremendous source of errors!

**Plane Poiseuille geometry:**  $h$  = plane distance,  $\mu(y) = \mu_0 \left( 1 \pm \varepsilon \left( \frac{6y^2}{h^2} - \frac{1}{2} \right) \right)$ ,  $\mu_0$  = mean value. For

$\varepsilon = 0.1$  then  $0.9\mu_0 < \mu < 1.1\mu_0$

(same for cylindrical Couette geometry with a similar form of  $\mu$ ).

**Notice that the maximum variation of  $\mu/\mu_0$  is never more than 10%!**

**Governing eqs:**

$$\left[ \mu(y) \left( 1 + \beta (u')^2 \right)^m u' \right]' = C, \quad u(-h/2) = u(h/2) = 0,$$

where  $C$  is the (negative) constant pressure gradient in the  $x$ -direction.

Define

$$Q = \int_{-h/2}^{h/2} u(y) dy \text{ (vol. flow rate)}, \quad Q_{\text{err}} = \left| \frac{Q - Q_H}{Q_H} \right|, \quad u'_{\text{err}} = \left| \frac{u' - u'_H}{u'_H} \right|$$

("H" identifies the correspondent power law fluid such that  $\mu(y) = \mu_0$ )

## (Anand & Rajagopal, 2005) Forgetting inhomogeneity can be a tremendous source of errors!

For  $m = 0.5$ ,  $C = 2.0$ ,  $\varepsilon = 0.1$ . Inhomogeneity given by  $\mu = \mu_0 \left( 1 + \varepsilon \left( \frac{6y^2}{h^2} - \frac{1}{2} \right) \right)$ .

**Maximum error (at the wall) is 97.77%.**

$m$	$C$	Variable	$\beta$			
			1.0	0.81	0.25	0.01
-0.5	-2.0	wall shear rate error	97.77%	31.07%	11.61%	9.17%
-0.5	-2.0	flow mass error	34.60%	11.98%	4.76%	3.74%
-0.4	-15.0	wall shear rate error	37.91%	37.90%	37.89%	13.90%
-0.4	-15.0	flow mass error	26.86%	26.86%	26.84%	5.75%

Table: Error in the flow rate and shear rate (at the wall) between the inhomogeneous fluid defined through

$\mu = \mu_0 \left( 1 + \varepsilon \left( \frac{6y^2}{h^2} - \frac{1}{2} \right) \right)$  ( $\varepsilon = 0.1$ ) and the corresponding homogeneous one.

## (Anand & Rajagopal, 2005) Forgetting inhomogeneity can be a tremendous source of errors!

But the situation can be even worse!

For  $m = 0.5$ ,  $C = 1.8$ ,  $\varepsilon = 0.1$ . Inhomogeneity given by  $\mu = \mu_0 \left( 1 - \varepsilon \left( \frac{6y^2}{h^2} - \frac{1}{2} \right) \right)$ .

**Maximum error (at the wall) is almost 600%.**

$m$	$C$	Variable	$\beta$			
			1.0	0.81	0.25	0.01
-0.5	-1.8	wall shear rate error	596%	49.48%	14.58%	11.22%
-0.5	-1.8	flow mass error	49.34%	14.17%	5.4%	4.38%
-0.4	-14.9	wall shear rate error	69.36%	69.34%	69.43%	20.12%
-0.4	-14.9	flow mass error	42.83%	42.71%	42.85%	7.45%

Table: Error in the flow rate and shear rate (at the wall) between the inhomogeneous fluid defined through

$\mu = \mu_0 \left( 1 - \varepsilon \left( \frac{6y^2}{h^2} - \frac{1}{2} \right) \right)$  ( $\varepsilon = 0.1$ ) and the corresponding homogeneous one.

## Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity**
- 3 The initial-boundary value problem
- 4 Numerics
- 5 Analytic solutions
- 6 Comparison between the analytic solution and the numerical solution

## Our framework and geometry

The examples by Anand and Rajagopal just emphasize the problem but are rather **artificial and unrealistic** since the viscosity  $\mu$  is explicitly assigned as a function of the spatial coordinates.

In a **real situation** we generally have

$$\mu = \mu(p, \varrho, \mathbb{D}^2)$$

i.e. is constitutively assigned through **unknown** quantities. Here we consider a simple case:

$$\psi = \psi(\varrho), \quad \mu = \mu(\varrho) \quad \Rightarrow \quad \mathbb{T} = -p\mathbb{I} + 2\mu(\varrho)\mathbb{D}$$

where  $\text{tr } \mathbb{D} = 0$  (i.e.  $\nabla \cdot \mathbf{u} = 0$ ), i.e. we have assumed the “incompressibility condition”.

The mass distribution of the material is not constant in the reference configuration but its motion can be considered *isochoric*, i.e. the density is constant along each particle path

## Our framework and geometry

The examples by Anand and Rajagopal just emphasize the problem but are rather **artificial and unrealistic** since the viscosity  $\mu$  is explicitly assigned as a function of the spatial coordinates.

In a **real situation** we generally have

$$\mu = \mu(p, \varrho, \mathbb{D}^2)$$

i.e. is constitutively assigned through **unknown** quantities. Here we consider a simple case:

$$\psi = \psi(\varrho), \quad \mu = \mu(\varrho) \quad \Rightarrow \quad \mathbb{T} = -p\mathbb{I} + 2\mu(\varrho)\mathbb{D}$$

where  $\text{tr } \mathbb{D} = 0$  (i.e.  $\nabla \cdot \mathbf{u} = 0$ ), i.e. we have assumed the “incompressibility condition”.

The mass distribution of the material is not constant in the reference configuration but its motion can be considered *isochoric*, i.e. the density is constant along each particle path

## Our framework and geometry

The examples by Anand and Rajagopal just emphasize the problem but are rather **artificial and unrealistic** since the viscosity  $\mu$  is explicitly assigned as a function of the spatial coordinates.

In a **real situation** we generally have

$$\mu = \mu(p, \varrho, \mathbb{D}^2)$$

i.e. is constitutively assigned through **unknown** quantities. Here we consider a simple case:

$$\psi = \psi(\varrho), \quad \mu = \mu(\varrho) \quad \Rightarrow \quad \mathbb{T} = -p\mathbb{I} + 2\mu(\varrho)\mathbb{D}$$

where  $\text{tr } \mathbb{D} = 0$  (i.e.  $\nabla \cdot \mathbf{u} = 0$ ), i.e. we have assumed the “incompressibility condition”.

The mass distribution of the material is not constant in the reference configuration but its motion can be considered *isochoric*, i.e. the density is constant along each particle path

## Our framework and geometry

The examples by Anand and Rajagopal just emphasize the problem but are rather **artificial and unrealistic** since the viscosity  $\mu$  is explicitly assigned as a function of the spatial coordinates.

In a **real situation** we generally have

$$\mu = \mu(p, \varrho, \mathbb{D}^2)$$

i.e. is constitutively assigned through **unknown** quantities. Here we consider a simple case:

$$\psi = \psi(\varrho), \quad \mu = \mu(\varrho) \quad \Rightarrow \quad \mathbb{T} = -p\mathbb{I} + 2\mu(\varrho)\mathbb{D}$$

where  $\text{tr } \mathbb{D} = 0$  (i.e.  $\nabla \cdot \mathbf{u} = 0$ ), i.e. we have assumed the “incompressibility condition”.

**The mass distribution of the material is not constant in the reference configuration but its motion can be considered *isochoric*, i.e. the density is constant along each particle path**

## Our framework and geometry

Consider a finite rigid container open at the endwalls in the  $x$ -direction and assume the “lubrication hypothesis”. The fluid is driven by a given horizontal pressure drop.

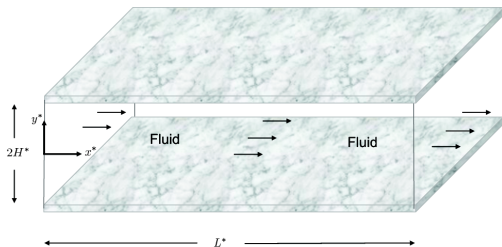


Figure 1. Sketch of the system.

- 1 Problem intrinsically 2D
- 2  $\frac{\text{channel width}}{\text{channel length}} \ll 1$   
 (“lubrication condition”)

### Main consequence

**At leading order the momentum equation can be solved analytically!**

## Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity
- 3 The initial-boundary value problem**
- 4 Numerics
- 5 Analytic solutions
- 6 Comparison between the analytic solution and the numerical solution

## Mathematical problem: balance equations

Mass balance

$$\frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla \varrho = 0$$

Balance of linear momentum (with zero body forces)

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}] \mathbf{v} \right) = -\nabla p + \nabla \cdot [\mu(\varrho) \mathbb{D}]$$

Incompressibility

$$\text{tr } \mathbb{D} = \nabla \cdot \mathbf{v} = 0$$

**Unknowns:** the velocity  $\mathbf{v}$ , the density  $\varrho$  and the pressure  $p$ .

## Mathematical problem: balance equations

Mass balance

$$\frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla \varrho = 0$$

Balance of linear momentum (with zero body forces)

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}] \mathbf{v} \right) = -\nabla p + \nabla \cdot [\mu(\varrho) \mathbb{D}]$$

Incompressibility

$$\text{tr } \mathbb{D} = \nabla \cdot \mathbf{v} = 0$$

**Unknowns:** the velocity  $\mathbf{v}$ , the density  $\varrho$  and the pressure  $p$ .

## Mathematical problem: balance equations

Mass balance

$$\frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla \varrho = 0$$

Balance of linear momentum (with zero body forces)

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}] \mathbf{v} \right) = -\nabla p + \nabla \cdot [\mu(\varrho) \mathbb{D}]$$

Incompressibility

$$\text{tr } \mathbb{D} = \nabla \cdot \mathbf{v} = 0$$

**Unknowns:** the velocity  $\mathbf{v}$ , the density  $\varrho$  and the pressure  $p$ .

## Mathematical problem: balance equations

Mass balance

$$\frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla \varrho = 0$$

Balance of linear momentum (with zero body forces)

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}] \mathbf{v} \right) = -\nabla p + \nabla \cdot [\mu(\varrho) \mathbb{D}]$$

Incompressibility

$$\text{tr } \mathbb{D} = \nabla \cdot \mathbf{v} = 0$$

**Unknowns:** the velocity  $\mathbf{v}$ , the density  $\varrho$  and the pressure  $p$ .

## Mathematical problem: balance equations

Mass balance

$$\frac{\partial \varrho}{\partial t} + \mathbf{v} \cdot \nabla \varrho = 0$$

Balance of linear momentum (with zero body forces)

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}] \mathbf{v} \right) = -\nabla p + \nabla \cdot [\mu(\varrho) \mathbb{D}]$$

Incompressibility

$$\text{tr } \mathbb{D} = \nabla \cdot \mathbf{v} = 0$$

**Unknowns:** the velocity  $\mathbf{v}$ , the density  $\varrho$  and the pressure  $p$ .

## Mathematical problem: initial and boundary conditions for $v$ and $p$

Set  $\mathbf{v}(x, y, t) = (u(x, y, t), v(x, y, t))$ ,  $p = p(x, y, t)$ .

Initial condition

$$\mathbf{v}(x, y, 0) = \mathbf{v}_s(x, y)$$

No slip at the upper wall

$$u(x, H, t) = v(x, H, t) = 0$$

Symmetry at  $y = 0$

$$\frac{\partial u}{\partial y}(x, 0, t) = v(x, 0, t) = 0$$

Pressure drop driving the flow

$$p(0, y, t) = \Delta p > 0, \quad p(L, y, t) = 0 \quad (\Delta p \text{ may depend on } t)$$

## Mathematical problem: initial and boundary conditions for $\varrho$

The density  $\varrho$  must satisfy a first order partial differential equation. Then we need

### Boundary conditions

$$\varrho(x, 0, t) = \varrho_m(x, t), \quad \varrho(0, y, t) = \varrho_\ell(y, t),$$

### Initial conditions

$$\varrho(x, y, 0) = \varrho_s(x, y)$$

where  $\varrho_m$  (density in the middle plane),  $\varrho_\ell$  (density at the lateral inlet),  $\varrho_s$  (initial density) are given functions.

### Compatibility conditions

$$\varrho_m(x, 0) = \varrho_s(x, 0), \quad \varrho_\ell(y, 0) = \varrho_s(0, y), \quad \varrho_m(0, t) = \varrho_\ell(0, t)$$

## Domain of definition of $\varrho_\ell$ , $\varrho_m$ , $\varrho_s$

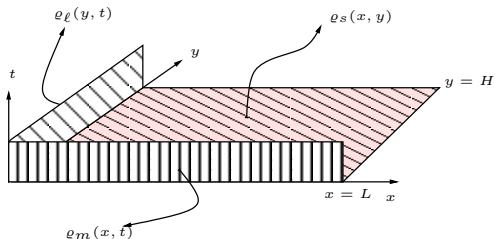


Figure: Boundary conditions for  $\varrho$

$\varrho_m$  cannot be assigned arbitrarily: symmetry and mass conservation imply

$$\frac{\partial \varrho_m(x, t)}{\partial t} + u(x, 0, t) \frac{\partial \varrho_m(x, t)}{\partial x} = 0$$

$$\varrho_m(0, t) = \varrho_\ell(0, t), \quad \varrho_m(x, 0) = \varrho_s(x, 0) \text{ (compatibility)}$$

## Domain of definition of $\varrho_\ell$ , $\varrho_m$ , $\varrho_s$

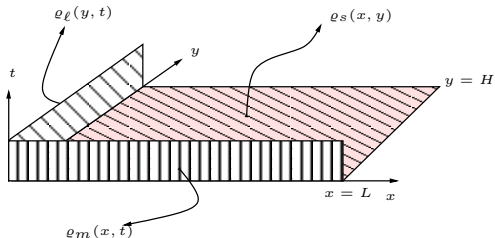


Figure: Boundary conditions for  $\varrho$

**$\varrho_m$  cannot be assigned arbitrarily:** symmetry and mass conservation imply

$$\frac{\partial \varrho_m(x, t)}{\partial t} + u(x, 0, t) \frac{\partial \varrho_m(x, t)}{\partial x} = 0$$

$$\varrho_m(0, t) = \varrho_\ell(0, t), \quad \varrho_m(x, 0) = \varrho_s(x, 0) \text{ (compatibility)}$$

## Dimensionless equation system to be solved

Choose suitable “scales” factor  $\rho_0, U, H, \mu_0$  to write equations in dimensionless form:

$$\left\{ \begin{array}{l} u_x + v_y = 0, \\ \rho_t + u\rho_x + v\rho_y = 0 \\ \varepsilon \text{Re} (u_t + uu_x + vv_y) = -p_x + \varepsilon^2 (2\mu u_x)_x + (\mu u_y + \varepsilon^2 \mu v_x)_y, \\ \varepsilon^3 (v_t + uv_x + vv_y) = -p_y + (\mu u_y + \varepsilon^2 \mu v_x)_x + \varepsilon^2 (2\mu v_y)_y. \end{array} \right.$$

where

$$\varepsilon = H/L, \quad \text{Re} = \frac{\rho_0 U H}{\mu_0}$$

are, respectively, the “aspect ratio” and the Reynolds number corresponding to the given characteristic scale factors.

## Lubrication approximation

Assuming  $\varepsilon \ll 1$ , at leading order in  $\varepsilon$  the system reduces to

$$\left\{ \begin{array}{l} u_x + v_y = 0, \\ \rho_t + u\rho_x + v\rho_y = 0 \\ 0 = -p_x + (\mu u_y)_y, \\ \mathbf{0} = -p_y. \end{array} \right.$$

Notice that  $p = p(x, t)$  (as usual in the lubrication theory).

At this stage, simple manipulations allow to obtain a **single (but not easy) equation for the density**

## Lubrication approximation

Assuming  $\varepsilon \ll 1$ , at leading order in  $\varepsilon$  the system reduces to

$$\begin{cases} u_x + v_y = 0, \\ \rho_t + u\rho_x + v\rho_y = 0 \\ 0 = -p_x + (\mu u_y)_y, \\ \mathbf{0} = -p_y. \end{cases}$$

Notice that  $p = p(x, t)$  (as usual in the lubrication theory).

At this stage, simple manipulations allow to obtain a **single (but not easy) equation for the density**

## A first order PDE whose coefficients are operators

$$(1) \quad \boxed{\varrho_t + \mathbf{u}(\varrho)\varrho_x + \mathbf{v}(\varrho)\varrho_y = 0}$$

where  $\mathbf{u}, \mathbf{v}$  are (formal) **operators** in  $\varrho$ : indeed  $\mu = \mu(\varrho(x, s, t))$  and

$$\mathbf{u} : \varrho \mapsto \frac{\Delta p}{\int_0^1 \frac{dx}{\int_0^1 dz \int_z^1 \frac{s}{\mu(\varrho)} ds}} \cdot \frac{\int_y^1 \frac{s}{\mu(\varrho)} ds}{\int_0^1 \int_z^1 \frac{s}{\mu(\varrho)} ds}$$

$$\mathbf{v} : \varrho \mapsto \frac{d}{dx} \left[ \frac{\Delta p}{\int_0^1 \frac{dx}{\int_0^1 dz \int_z^1 \frac{s}{\mu(\varrho)} ds}} \cdot \frac{\int_y^1 dz \int_z^1 \frac{s}{\mu(\varrho)} ds}{\int_0^1 \int_z^1 \frac{s}{\mu(\varrho)} ds} \right]$$

## Complete system to be solved

$$(2) \quad \left\{ \begin{array}{l} (\varrho_m)_t + [u(\varrho)]|_{y=0} (\varrho_m)_x = 0, \\ \varrho_m(x, 0) = \varrho_s(x, 0), \\ \varrho_m(0, t) = \varrho_\ell(0, t), \\ \varrho_t + u(\varrho)\varrho_x + v(\varrho)\varrho_y = 0, \\ \varrho(x, 0, t) = \varrho_m(x, t), \\ \varrho(0, y, t) = \varrho_\ell(y, t), \\ \varrho(x, y, 0) = \varrho_s(x, y), \end{array} \right.$$

with the compatibility condition

$$(3) \quad \varrho_s(0, y) = \varrho_\ell(y, 0).$$

In conclusion we have to solve a system of **two first order pde's for the unknowns**  $\varrho_m(x, t)$  **and**  $\varrho(x, y, t)$ .

## Check physical consistency in simple cases

Case I: Suppose

$$\varrho_\ell(0, t) = \varrho_s(x, 0) = 1.$$

Then  $\varrho_m(x, t) \equiv 1$  is the only solution of (2)<sub>1</sub> and IBV for (1) reduces to

$$(4) \quad \left\{ \begin{array}{l} \frac{\partial \varrho}{\partial t} + u(\varrho) \frac{\partial \varrho}{\partial x} = 0, \\ \varrho(x, 0, t) = 1, \\ \varrho(0, y, t) = \varrho_\ell(y, t), \\ \varrho(x, y, 0) = \varrho_s(x, y), \end{array} \right.$$

## Check physical consistency in simple cases

Case II: Suppose

$$\varrho_\ell(y, t) = \varrho_s(x, y) = 1.$$

Then system (2) is automatically satisfied by  $\varrho_m(x, t) = \varrho(x, y, t) \equiv 1$ . In this case  $\mu$  is a constant and, taking  $\mu = 1$ , it follows

$$u = \Delta p \left( \frac{1 - y^2}{2} \right), \quad v = 0,$$

i.e. the **classical parabolic profile**.

## Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity
- 3 The initial-boundary value problem
- 4 Numerics**
- 5 Analytic solutions
- 6 Comparison between the analytic solution and the numerical solution

## The method

Pick a function  $\varrho^{(0)}(x, y, t)$  representing a “guess” for the solution of problem (2) in some time interval  $[0, t]$  and evaluate the coefficients  $u(\varrho^{(0)}), v(\varrho^{(0)}), u(\varrho^{(0)})|_{y=0}$  appearing in (2)<sub>1,4</sub>. Then we solve the problem (2)<sub>1-3</sub> (which has now become **linear**) and use the solution as boundary condition in (2)<sub>5</sub>. Finally we solve the problem (2)<sub>4-7</sub> getting a solution  $\varrho^{(1)}(x, y, t)$  that clearly depends on the particular choice of  $\varrho^{(0)}(x, y, t)$ . With this procedure we define the map

$$\Psi : \varrho^{(0)}(x, y, t) \mapsto \varrho^{(1)}(x, y, t)$$

We then use  $\varrho^{(1)}(x, y, t)$  as the new guess and determine  $\varrho^{(2)}(x, y, t) = \Psi(\varrho^{(1)})$ .

## The method

Proceeding in this way, we build a sequence of functions

$$\left\{ \varrho^{(n)}(x, y, t) \right\}, \quad n \in \mathbb{N}$$

If the map  $\Psi$  has a unique fixed point (a proof is still pending) in some function space then the unique fixed point is the solution to problem (2).

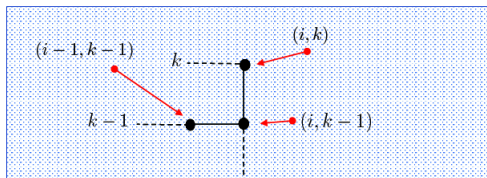
The iterative procedure will be stopped when the quantity

$$\text{tol} = \left\| \varrho^{(n+1)} - \varrho^{(n)} \right\| = \sup \left| \varrho^{(n+1)} - \varrho^{(n)} \right|$$

is less than a fixed tolerance.

## Finite difference schemes for $\rho_m$ and $\rho$

(a) 2D stencil  
problem for  $\rho_m$



(b) 3D stencil  
problem for  $\rho$

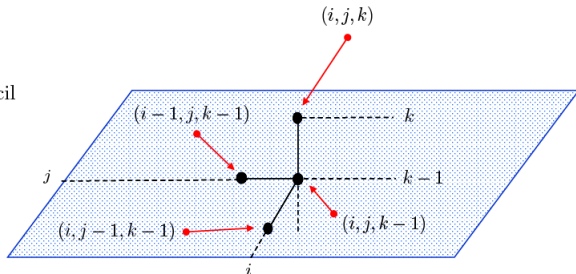


Figure 2. Stencil for the 2D and 3D problems.

**Example I:**  $\Delta p = 1$ ,  $\mu(\varrho) = \exp(\varrho - 1)$

$$\varrho_s(x, y) = 1 + \frac{4}{5} \sin(6x^2) + \frac{4}{5} \cos(4y^2), \varrho_t(y, t) = 1 + \frac{4}{5} \cos(4y^2)$$

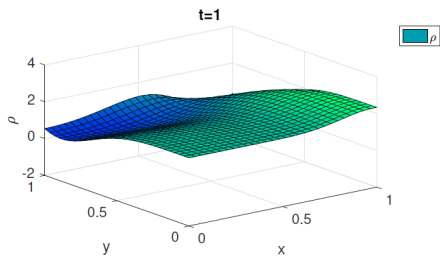
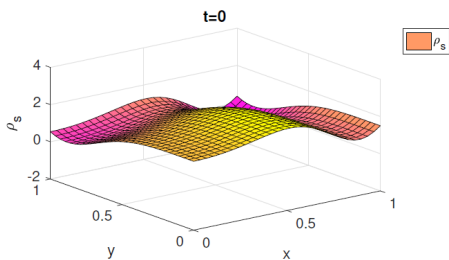


Figure:  $\varrho_s(x, y)$  and  $\varrho(x, y, 1)$

**Example II:**  $\Delta p = 1$ ,  $\mu(\varrho) = \exp(\varrho - 1)$

$$\varrho_s(x, y) = \exp \left[ -4 \left( x - \frac{1}{2} \right)^2 - 4y^2 \right] + \frac{x}{x^3 + 2}, \quad \varrho_\ell(y, t) = \exp \left[ -1 - y^2 \right]$$

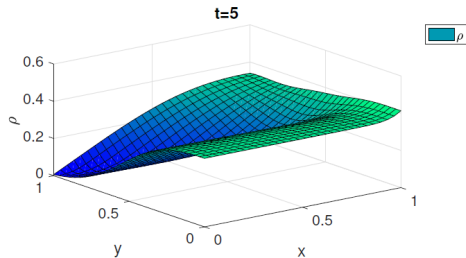
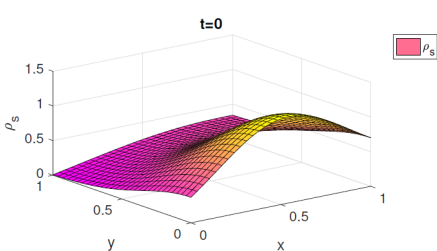


Figure:  $\varrho_s(x, y)$  and  $\varrho(x, y, 5)$

## Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity
- 3 The initial-boundary value problem
- 4 Numerics
- 5 Analytic solutions**
- 6 Comparison between the analytic solution and the numerical solution

## Small perturbation of the trivial solution

Recall that  $\varrho \equiv 1$  automatically satisfies the system (2) and leads to the **classical Navier-Stokes system**. Therefore take a **small perturbation** of this very special case:

$$\varrho = 1 + \delta\zeta(x, y, t), \quad \delta \ll 1.$$

Expand  $\mu$  around  $\varrho = 1$ :

$$\mu(\varrho) = \mu(1) + \mu'(1)(\varrho - 1) + o(\varrho - 1)$$

We are dealing with non dimensional variables so we can take  $\mu(1) = 1$  without loss of generality.

Thus at the first order

$$(5) \quad \mu(\varrho) = 1 + \delta\beta(\varrho - 1)$$

with  $\beta = \mu'(1)$ . Use now (5) to evaluate  $u(\varrho)$  and  $v(\varrho)$

## First order expansions of $u(\varrho)$ and $v(\varrho)$

$$u = u_0 + \delta u_1, \quad v = \delta v_1$$

with  $u_0 = \Delta p (1 - y^2) / 2$ ,  $v_0 = 0$  (the Navier-Stokes elementary solution) and

$$(6) \quad u_1 = \Delta p \left\{ \left( \frac{1 - y^2}{2} \right) \left[ 3 \int_0^1 \int_z^1 s \beta \zeta \right] - \int_y^1 s \beta \zeta \right\},$$

$$(7) \quad v_1 = \Delta p \left\{ \left( \frac{(1 - y^2)(y + 2)}{2} \right) \left[ 3 \int_0^1 \int_z^1 s \beta \zeta - 3 \int_0^1 \int_0^1 \int_z^1 s \beta \zeta \right] - \int_y^1 \int_z^1 s \beta \zeta \right\}.$$

Finally use (5),(6),(7) into (2).

The **zeroth order** problem, i.e. the one in which all the terms containing  $\delta$  are neglected, **is (as expected) the classical Navier-Stokes system with constant density.**

## The first order problem w.r.t. $\delta$

$$(8) \quad \begin{cases} \zeta_t + \Delta p \left( \frac{1-y^2}{2} \right) \zeta_x = 0, & (\zeta_m)_t + \frac{\Delta p}{2} (\zeta_m)_x = 0, \\ \zeta_m(x, 0) = \zeta_s(x, 0), & \zeta_m(0, t) = \zeta_\ell(0, t), & \zeta(x, 0, t) = \zeta_m(x, t), \\ \zeta(0, y, t) = \zeta_\ell(y, t), & \zeta(x, y, 0) = \zeta_s(x, y), \end{cases}$$

where

$$\varrho_\ell(y, t) = 1 + \delta \zeta_\ell(y, t), \quad \varrho_s(x, y) = 1 + \delta \zeta_s(x, y)$$

and  $\zeta_\ell(y, 0) = \zeta_s(0, y)$  (compatibility).

Notice that  $y$  in (8)<sub>1</sub> plays the role of a parameter and that (8)<sub>2,3,4</sub> for  $\zeta_m$  can be solved by the method of characteristics (MoC): **solution is**

$$\zeta_m(x, t) = \bar{\zeta}_s \left( x - \frac{\Delta p}{2} t, 0 \right), \quad \bar{\zeta}_s = \begin{cases} \zeta_s(x, y), & x \geq 0 \\ \zeta_\ell \left( y - \frac{2x}{\Delta p(1-y^2)} \right), & x \leq 0 \end{cases}$$

## The first order problem w.r.t. $\delta$

We are left now with

$$(9) \quad \left\{ \begin{array}{l} \zeta_t + \Delta p \left( \frac{1-y^2}{2} \right) \zeta_x = 0, \\ \zeta(x, 0, t) = \bar{\zeta}_s \left( x - \frac{\Delta p}{2} t, 0 \right), \\ \zeta(0, y, t) = \zeta_\ell(y, t), \\ \zeta(x, y, 0) = \zeta_s(x, y). \end{array} \right.$$

**Solution** (by MoC))

$$(10) \quad \zeta(x, y, t) = \bar{\zeta}_s \left( x - \frac{\Delta p}{2} (1-y^2) t, y \right)$$

## Table of Contents

- 1 Homogeneous vs. inhomogeneous
- 2 Thin film flow of an inhomogeneous fluid with density dependent viscosity
- 3 The initial-boundary value problem
- 4 Numerics
- 5 Analytic solutions
- 6 Comparison between the analytic solution and the numerical solution**

## Comparison between the analytic solution and the numerical solution

Consider the following pair of boundary conditions

$$(11) \quad \begin{cases} \varrho_s(x, y) = 1 + \delta \left[ \sin(6x^2) + \frac{4}{5} \cos(4y^2) \right], \\ \varrho_\ell(y, t) = 1 + \delta [\cos(4y^2)] \end{cases}$$

and solve the problem both with the perturbation approach and with the numerical scheme: let  $\varrho_p(x, y, t)$  the *perturbation analytic solution* and  $\varrho_u(x, y, t)$  the *solution obtained with the numerical approach*. Define

$$\theta = \varrho_p - \varrho_u$$

and use  $\delta$  as a parameter.

## Comparison between the analytic solution and the numerical solution

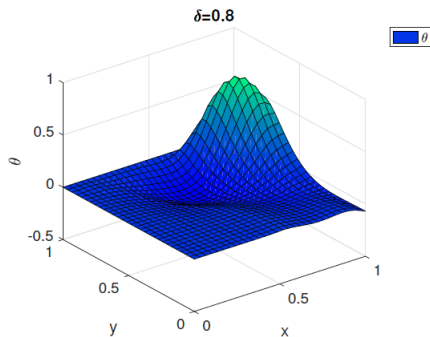


Figure: Function  $\theta$  for decreasing  $\delta$

## Comparison between the analytic solution and the numerical solution

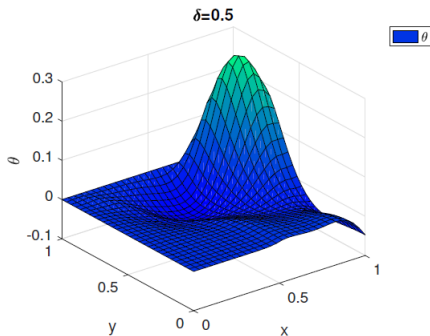


Figure: Function  $\theta$  for decreasing  $\delta$

## Comparison between the analytic solution and the numerical solution

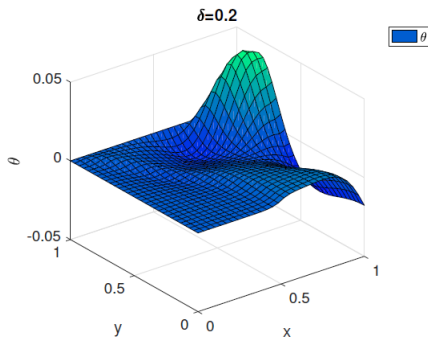


Figure: Function  $\theta$  for decreasing  $\delta$

## Comparison between the analytic solution and the numerical solution

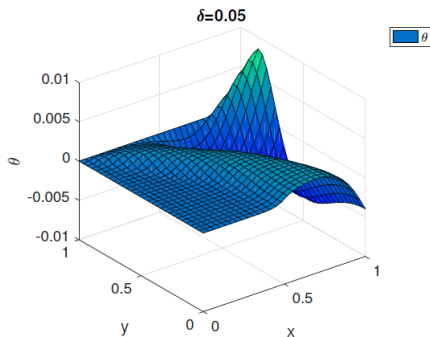


Figure: Function  $\theta$  for decreasing  $\delta$

## Results

- Perturbation approach is advantageous when the boundary data are small perturbations of the homogeneous case.
- Comparison shows that the absolute error  $\theta$  decreases for decreasing  $\delta$  as expected.
- The good agreement in the comparison does not only prove that the perturbation approach is useful for small density variation around the constant value (homogeneous system), but also that the numerical method is reliable and can be used to infer the density distribution with general boundary data.



Agradesco a todos los presentes por la atención prestada.  
Muchas gracias